Dynamics of radiation due to vacuum nonlinearities

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In quantum electrodynamics, photon–photon scattering can be the result of the exchange of virtual electron–positron pairs. This gives rise to a non-trivial dispersion relation for a single photon moving on a background of electromagnetic fields. Knowledge of the dispersion relation can be transferred, using standard methods, into new insights in the dynamical equations for the photons. Effectively, those equations will contain different types of self-interaction terms, depending on whether the photons are coherent or not. It is shown that coherent photons are governed by a nonlinear Schrödinger type equation, such that the self-interaction terms vanish in the limit of parallel propagating waves. For incoherent photons, a set of fluid equations can determine the evolution of the corresponding radiation gas. In the case of a self-interacting radiation fluid, it is shown that Landau damping can occur.

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I. INTRODUCTION

According to QED, the non-classical phenomenon of photon–photon scattering can take place due to the exchange of virtual electron–positron pairs. This is a second order effect (in terms of the fine structure constant $\alpha \equiv e^2/4\pi\varepsilon_0\hbar c \approx 1/137$), which in standard notations can be deduced from the Euler–Heisenberg Lagrangian density [1, 2]

$$\mathscr{L} = \varepsilon_0 \mathscr{F} + \varepsilon_0^2 \kappa (4\mathscr{F}^2 + 7\mathscr{G}^2) \ , \tag{1}$$

where $\kappa \equiv 2\alpha^2\hbar^3/45m_e^4c^5$, $\mathscr{F} \equiv \frac{1}{2}(E^2 - c^2B^2)$, $\mathscr{G} \equiv c\mathbf{E}\cdot\mathbf{B}$, e is the electron charge, c the velocity of light, $2\pi\hbar$ the Planck constant and m_e the electron mass. The Lagrangian (1) is valid as long as there is no pair creation and the field strength is smaller than the critical field, i.e.

$$\omega \ll m_e c^2/\hbar \quad \text{and} \quad |\boldsymbol{E}| \ll E_{\text{crit}} \equiv m_e c^2/e\lambda_c \tag{2}$$

respectively. Here λ_c is the Compton wave length, and $E_{\text{crit}} \simeq 10^{18}\,\text{V/m}$. The latter terms in (1) represent the effects of vacuum polarisation and magnetisation. We note that $\mathscr{F} = \mathscr{G} = 0$ in the limit of parallel propagating waves. It is therefore necessary to use other waves in order to obtain an effect from the QED corrections. Several attempts have been presented in the literature [3, 4, 5, 6, 7, 8, 9], where Refs. [3, 4, 5, 6] mainly focused on principal issues, whereas the experimental possibilities for detection have been discussed in Refs. [7, 8, 9].

The non-trivial propagation of photons in strong background electromagnetic fields, due to effects of nonlinear electrodynamics, has been considered in a number of papers (see, e.g., Refs. [11, 12] and references therein). Their main focus was on the interesting effects of photon splitting and birefringence in vacuum. However, Thoma [13] investigated the interaction of photons with a photon gas, using the real-time formalism, and calculated the corresponding change in the speed of light due to the Cosmic Microwave Background (CMB).

In the present paper, we derive an evolution equation for an ensemble of electromagnetic pulses. Moreover, the methods of radiation hydrodynamics [14] are combined with the QED theory for photon–photon scattering, and a system of equations (c.f. [15]) is obtained, where the radiation pressure of the pulse will act as a driver of acoustic waves in the photon gas.

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II. EQUATIONS AND EXAMPLES

In a medium with polarisation $\mathbf{P}$ and magnetisation $\mathbf{M}$ the general wave equations for $\mathbf{E}$ and $\mathbf{B}$ are

$$\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} - \nabla^2\mathbf{E} = -\mu_0\left[\frac{\partial^2\mathbf{P}}{\partial t^2} + c^2\nabla(\nabla\cdot\mathbf{P}) + \frac{\partial}{\partial t}(\nabla\times\mathbf{M})\right] , \tag{3a}$$

and

$$\frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} - \nabla^2\mathbf{B} = \mu_0\left[\nabla\times(\nabla\times\mathbf{M}) + \frac{\partial}{\partial t}(\nabla\times\mathbf{P})\right] . \tag{3b}$$

Using the Lagrangian (1), one can show that due to photon-photon scattering induced by the exchange of virtual electron-positron pairs, the vacuum will effectively be polarised and magnetised, and the polarisation and magnetisation will be nonlinear to cubic order in the electromagnetic field (see, e.g., Ref. [7]). These equations can be used to analyse the properties of coherent radiation, and it is straightforward to show that the non-linearities vanish in the case of co-propagating plane waves. On the other hand, introducing dispersion into the propagation, by the use of wave guides, will give non-trivial polarisations and magnetisation, and this has been used as a suggested means of detection of photon–photon scattering [9], and to form light bullets in vacuum [10].

On the other hand, as shown in Ref. [11], one can derive the dispersion relation for a single photon moving on a given background of electromagnetic fields, using first principles. The result is (see also Ref. [12] and references therein)

$$\omega(\boldsymbol{k},\boldsymbol{E},\boldsymbol{B}) = c|\boldsymbol{k}|\left(1 - \tfrac{1}{2}\lambda|\boldsymbol{Q}|^2\right) . \tag{4}$$

where

$$|\boldsymbol{Q}|^2 = \varepsilon_0\left[E^2 + c^2B^2 - (\hat{\boldsymbol{k}}\cdot\boldsymbol{E})^2 - c^2(\hat{\boldsymbol{k}}\cdot\boldsymbol{B})^2 - 2c\hat{\boldsymbol{k}}\cdot(\boldsymbol{E}\times\boldsymbol{B})\right] , \tag{5}$$

and $\lambda = \lambda_\pm$, where $\lambda_+ = 14\kappa$ and $\lambda_- = 8\kappa$ for the two different polarisation states of the photon. Furthermore, $\hat{\boldsymbol{k}} \equiv \boldsymbol{k}/k$. The approximation $\lambda|\boldsymbol{Q}|^2 \ll 1$ has been used. The background electric and magnetic fields are denoted by $\boldsymbol{E}$ and $\boldsymbol{B}$, respectively.

From expressions (4) and (5) we can derive essentially different dispersion relations, depending on whether or not the background is coherent or incoherent.

To start with, suppose a photon with wave vector $\boldsymbol{k}$ is moving on a background of photons close to thermal equilibrium. The background electromagnetic fields then satisfy $\langle\boldsymbol{E}\rangle = \langle\boldsymbol{B}\rangle = 0$, $\langle E_iE_j\rangle = \langle E^2\rangle\delta_{ij}/3$, $\langle B_iB_j\rangle = \langle B^2\rangle\delta_{ij}/3$, where the angular brackets denote ensemble average. From (5) we then obtain

$$|\boldsymbol{Q}_g|^2 = \tfrac{4}{3}\mathscr{E}_g, \tag{6}$$

where $\mathscr{E} = \varepsilon_0(\langle E^2\rangle + c^2\langle B^2\rangle)/2$ is the energy density of the radiation gas.

On the other hand, if the photon propagates on a plane wave background, such that $\boldsymbol{E} = E\hat{\boldsymbol{e}}$, and $\boldsymbol{B} = E\hat{\boldsymbol{k}}_p\times\hat{\boldsymbol{e}}/c$, where $\hat{\boldsymbol{e}}$ is the unit electric vector and $\boldsymbol{k}_p$ is the wave vector of the background field, we find that

$$|\boldsymbol{Q}|^2 = \left[2 - 2(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{k}}_p) - (\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{e}})^2 - (\hat{\boldsymbol{k}}\cdot(\hat{\boldsymbol{k}}_p\times\hat{\boldsymbol{e}}))^2\right]\varepsilon_0|E|^2. \tag{7}$$

If $\boldsymbol{k} = \boldsymbol{k}_p$, the interaction term $|\boldsymbol{Q}|^2$ vanishes identically, consistent with the previous statement that a plane wave pulse can not self-interact. On the other hand, if the single photon belongs to a random ensemble of photons, i.e., a photon gas close to thermal equilibrium, we have $\langle\hat{\boldsymbol{k}}\rangle = 0$ and $\langle\hat{k}_i\hat{k}_j\rangle\delta_{ij}/3$. The expression (7) then reduces to

$$|\boldsymbol{Q}_p|^2 = \tfrac{4}{3}\mathscr{E}_p, \tag{8}$$

where $\mathscr{E}_p = \varepsilon_0|E|^2$ is the energy density of the background electromagnetic pulse.

Using the dispersion relation above, different scenarios can be analysed.

A. An ensemble of pulses

For a plane wave pulse moving on a background of photons, randomly distributed in momentum space, the relation (6) holds. It is then straightforward to derive, by using the eikonal representation and the WKBJ approximation [15], the Schrödinger equation [16]

$$i\left(\frac{\partial}{\partial t}+c\hat{\boldsymbol{k}}_0\cdot\nabla\right)E+\frac{c}{2k_0}\left[\nabla^2-(\hat{\boldsymbol{k}}_0\cdot\nabla)^2\right]E+\frac{2}{3}\lambda ck_0\mathscr{E}_gE=0, \tag{9}$$

where $\boldsymbol{k}_0$ is the linear vacuum wave number of the pulse. Suppose now that we have an ensemble of pulses. We decompose the electric field according to $E=\langle E\rangle+\widetilde{E}$, where $\langle\widetilde{E}\rangle=0$. Taking the average of Eq. (9), we obtain

$$i\left(\frac{\langle k_0\rangle}{c}\frac{\partial}{\partial t}+\langle k_{0i}\rangle\frac{\partial}{\partial x_i}\right)\langle E\rangle+\frac{1}{2}\left(\delta_{ij}-\langle\hat{k}_{0i}\hat{k}_{0j}\rangle\right)\frac{\partial^2\langle E\rangle}{\partial x_i\partial x_j}+\frac{2}{3}\lambda\langle k_0^2\rangle\mathscr{E}\langle E\rangle=\left\langle -i\frac{k_0}{c}\frac{\partial\widetilde{E}}{\partial t}-ik_{0i}\frac{\partial\widetilde{E}}{\partial x_i}+\frac{1}{2}\hat{k}_{0i}\hat{k}_{0j}\frac{\partial^2\widetilde{E}}{\partial x_i\partial x_j}\right\rangle, \tag{10}$$

where we have employed Einstein's summation convention. In general, we can subtract Eq. (10) from (9) in order to obtain a similar equation for $\widetilde{E}$. This would then constitute a general coupled system between the fluctuations and the average wave packet. We will not pursue this issue further, however.

We now write $\hat{\boldsymbol{k}}_0=\langle\hat{\boldsymbol{k}}_0\rangle+\tilde{\hat{\boldsymbol{k}}}_0$ with $\langle\tilde{\hat{\boldsymbol{k}}}_0\rangle=0$, which implies $\langle\hat{k}_{0i}\hat{k}_{0j}\rangle=\langle\hat{k}_{0i}\rangle\langle\hat{k}_{0j}\rangle+\langle\tilde{\hat{k}}_{0i}\tilde{\hat{k}}_{0j}\rangle$. If $\tilde{\hat{\boldsymbol{k}}}_0$ is approximately isotropically distributed, we then have $\langle\tilde{\hat{k}}_{0i}\tilde{\hat{k}}_{0j}\rangle=\frac{1}{3}\langle|\tilde{\hat{\boldsymbol{k}}}_0|^2\rangle\delta_{ij}$. Furthermore, since $|\hat{\boldsymbol{k}}_0|^2=1$, it follows that $0\leq\langle|\tilde{\hat{\boldsymbol{k}}}_0|^2\rangle=1-|\langle\hat{\boldsymbol{k}}_0\rangle|^2\leq 1$. Thus,

$$\langle\hat{k}_{0i}\hat{k}_{0j}\rangle=\langle\hat{k}_{0i}\rangle\langle\hat{k}_{0j}\rangle+\tfrac{1}{3}\left(1-|\langle\hat{\boldsymbol{k}}_0\rangle|^2\right)\delta_{ij}. \tag{11}$$

In the case of small random fluctuations in the field amplitude, i.e., $|\widetilde{E}|\ll|\langle E\rangle|$, we neglect the right hand side of Eq. (10). Suppose furthermore that the averaged wave vector, its direction, and its amplitude all change slowly over time and space compared to the amplitude $\langle E\rangle$. Moreover, for a self-interacting pulse ensemble, we see that $\mathscr{E}=\varepsilon_0\langle|E|^2\rangle\approx\varepsilon_0|\langle E\rangle|^2$. Using the relation (11), we thus obtain the dynamical equation

$$i\left(\frac{\partial}{\partial t}+\frac{c}{\langle k_0\rangle}\langle\boldsymbol{k}_0\rangle\cdot\nabla\right)\langle E\rangle+\frac{c}{2\langle k_0\rangle}\left[\frac{2}{3}\left(1+\frac{1}{2}|\langle\hat{\boldsymbol{k}}_0\rangle|^2\right)\nabla_\perp^2+\frac{2}{3}\left(1-|\langle\hat{\boldsymbol{k}}_0\rangle|^2\right)\nabla_{||}^2\right]\langle E\rangle+\frac{2}{3}\frac{\lambda c\varepsilon_0\langle k_0^2\rangle}{\langle k_0\rangle}|\langle E\rangle|^2\langle E\rangle=0, \tag{12}$$

where $\nabla_\perp^2=\nabla^2-\nabla_{||}^2$ and $\nabla_{||}^2=(\langle\hat{k}_{0i}\rangle\langle\hat{k}_{0j}\rangle/|\langle\hat{\boldsymbol{k}}_0\rangle|^2)\partial_i\partial_j$. Equation (12) therefore describes the dynamics of the non-fluctuating part of an ensemble of pulses, when the nonlinear self-interaction due to photon–photon scattering is taken into account.

B. The kinetic theory of non-linear photons

In general, from our dispersion relation one can formulate the single-particle dynamics in terms of the Hamiltonian ray equations. For a dispersion relation $\omega=ck[1-(\lambda/2)|\boldsymbol{Q}|^2]$, where $|\boldsymbol{Q}|^2$ is assumed to be independent of $\boldsymbol{k}$, we have the Hamiltonian ray equations

$$\dot{\boldsymbol{r}}=\frac{\partial\omega}{\partial\boldsymbol{k}}=c\left(1-\frac{1}{2}\lambda|\boldsymbol{Q}|^2\right)\hat{\boldsymbol{k}}, \tag{13a}$$

$$\dot{\boldsymbol{k}}=-\frac{\partial\omega}{\partial\boldsymbol{r}}=\frac{1}{2}\lambda ck\frac{\partial|\boldsymbol{Q}|^2}{\partial\boldsymbol{r}}, \tag{13b}$$

where $\dot{\boldsymbol{r}}$ is the group velocity of the photon, $\dot{\boldsymbol{k}}$ is the force on a photon, and the dot denotes time derivative.

The equation for the collective interaction of photons can then be formulated as [16, 17]

$$\frac{\partial N}{\partial t}+c\left(1-\frac{1}{2}\lambda|\boldsymbol{Q}|^2\right)\hat{\boldsymbol{k}}\cdot\frac{\partial N}{\partial\boldsymbol{r}}+\frac{1}{2}\lambda ck\frac{\partial|\boldsymbol{Q}|^2}{\partial\boldsymbol{r}}\cdot\frac{\partial N}{\partial\boldsymbol{k}}=0. \tag{14}$$

where the distribution function $N=N(\boldsymbol{k},\boldsymbol{r},t)$ has been normalised such that the number density is $n(\boldsymbol{r},t)=\int N(\boldsymbol{k},\boldsymbol{r},t)\,d\boldsymbol{k}$.

From Eq. (14) a hierarchy of fluid equations can be built [16], of which the two first are the energy conservation equation

$$\frac{\partial \mathscr{E}_g}{\partial t} + \nabla \cdot (\mathscr{E}_g \boldsymbol{u} + \boldsymbol{q}) = -\frac{1}{2}\lambda \mathscr{E} \frac{\partial |\boldsymbol{Q}|^2}{\partial t}, \tag{15a}$$

and the momentum conservation equation

$$\frac{\partial \boldsymbol{\Pi}}{\partial t} + \nabla \cdot \Big[\boldsymbol{u} \otimes \boldsymbol{\Pi} + \mathsf{P} \Big] = \frac{1}{2}\lambda \mathscr{E} \nabla |\boldsymbol{Q}|^2, \tag{15b}$$

respectively. Here

$$\mathscr{E}_g(\boldsymbol{r}, t) = \int \hbar\omega N \, d\boldsymbol{k} \tag{16}$$

is the energy density, $\boldsymbol{q}(\boldsymbol{r}, t) = \int \hbar\omega \boldsymbol{w} N \, d\boldsymbol{k}$ the energy flux, and we have made the split $\dot{\boldsymbol{r}} = \boldsymbol{u} + \boldsymbol{w}$, where $\langle \boldsymbol{w} \rangle = 0$. Furthermore, $\boldsymbol{\Pi} = \int \hbar \boldsymbol{k} N \, d\boldsymbol{k}$ is the momentum density, and $\mathsf{P} = \int \boldsymbol{w} \otimes (\hbar \boldsymbol{k}) N \, d\boldsymbol{k}$ is the pressure tensor.

C. The interaction between a radiation gas and an electromagnetic pulse

From Eqs. (6) and (7) it is clear that even though the interaction of a pulse with itself will vanish, this is not the case if the pulse moves on a background of incoherent photons, i.e., a radiation gas. The problem of an interacting pulse and an incoherent radiation gas via photon–photon scattering was investigated in Ref. [16], where a coupled system of equations was derived [15].

D. The self-interacting radiation gas

From (6) it follows that a kinetic photon gas may interact with itself via a ponderomotive force $\propto \nabla \mathscr{E}_g$. Using Eqs. (6) and (14), the kinetic equation for this type of gas is

$$\frac{\partial N}{\partial t} + c\left(1 - \frac{2}{3}\lambda\mathscr{E}\right) \hat{\boldsymbol{k}} \cdot \frac{\partial N}{\partial \boldsymbol{r}} + \frac{2}{3} ck\lambda \frac{\partial \mathscr{E}}{\partial \boldsymbol{r}} \cdot \frac{\partial N}{\partial \boldsymbol{k}} = 0. \tag{17}$$

Assuming $N(\boldsymbol{k}, \boldsymbol{r}, t) = N_0(\boldsymbol{k}) + N_1(\boldsymbol{k}) \exp[i(\boldsymbol{K} \cdot \boldsymbol{r} - \Omega t)]$, where $N_1 \ll N_0$ and $\mathscr{E}(\boldsymbol{r}, t) = \mathscr{E}_0 + \mathscr{E}_1 \exp[i(\boldsymbol{K} \cdot \boldsymbol{r} - \Omega t)]$, where $\mathscr{E}_1 \ll \mathscr{E}_0$, we linearise Eqs. (17) and (16) to obtain the dispersion relation

$$1 = \frac{2}{3}\lambda \hbar c^2 \int \frac{k^2 \, \boldsymbol{K} \cdot (\partial N_0 / \partial \boldsymbol{k})}{\Omega - c\boldsymbol{K} \cdot \hat{\boldsymbol{k}}} \, d\boldsymbol{k}. \tag{18}$$

If the background distribution is isotropic, i.e., $N_0(\boldsymbol{k}) = N_0(k)$, and introducing spherical coordinates in $\boldsymbol{k}$-space, we rewrite Eq. (18) as

$$1 = \frac{4\pi}{3}\lambda\hbar c \int_{-1}^{1} \frac{\alpha}{\beta - \alpha} \, d\alpha \int_0^\infty k^4 \frac{dN_0}{dk} \, dk \tag{19}$$

where $\beta = \Omega / cK$ and $\alpha = \hat{\boldsymbol{k}} \cdot \boldsymbol{K}/K$. Performing the angular integration in α, for $\beta < 1$, we obtain

$$1 = \frac{8\pi}{3}\lambda\hbar c \left[-1 + \beta \,\mathrm{arctanh}\,(\beta) - \pi i \beta\right] \int_0^\infty k^4 \frac{dN_0}{dk} \, dk. \tag{20}$$

We now assume that we have a Gaussian background distribution

$$N_0(k) = \frac{\mathscr{E}_0}{8\pi a^4 c\hbar} \exp\left[-\frac{k^2}{2a^2}\right], \tag{21}$$

where a is the width of the distribution. The dispersion relation (20) is then

$$1 = \frac{8}{3}\lambda\mathscr{E}_0 \left[-1 + \beta \,\mathrm{arctanh}\,(\beta) - \pi i \beta\right]. \tag{22}$$

III. CONCLUSION

In the present paper we have considered the interaction, due to the quantum electrodynamic photon-photon scattering, of electromagnetic waves and incoherent photons. For this purpose we have applied the dispersion relation derived in Ref. [11]. The dynamical equations for an ensemble of pulses has been derived, and a set of equations describing a radiation fluid has been presented. An expression showing Landau damping within the self-interacting radiation gas has been derived.

[1] Heisenberg W. and Euler H., Z. Physik **98**, 714 (1936).
[2] Schwinger J., Phys. Rev. **82**, 664 (1951).
[3] Valluri S. R. and Bhartia P., Can. J. Phys. **58**, 116 (1980).
[4] Rozanov, N. N., Zh. Eksp. Teor. Fiz. **113**, 513 (1998) (Sov. Phys. JETP **86**, 284 (1998)).
[5] Rozanov, N. N., Zh. Eksp. Teor. Fiz. **103**, 1996 (1993) (Sov. Phys. JETP **76**, 991 (1998)).
[6] Alexandrov, E. B., Anselm A. A. and Moskalev A. N., Zh. Eksp. Teor. Fiz. **89**, 1985 (1993) (Sov. Phys. JETP **62**, 680 (1985)).
[7] Soljacic M. and Segev M., Phys. Rev. A **62**, 043817 (2000).
[8] Ding Y. J. and Kaplan A. E., Phys. Rev. Lett. **63**, 2725 (1989).
[9] G. Brodin, M. Marklund and L. Stenflo, Phys. Rev. Lett. **87** 171801 (2001).
[10] G. Brodin, L. Stenflo, D. Anderson, M. Lisak, M. Marklund and P. Johannisson, Phys. Lett. A **306** 206 (2003).
[11] Z. Bialynicka–Birula and I. Bialynicki–Birula, Phys. Rev. D **2** 2341 (1970).
[12] G. Boillat, J. Math. Phys. **11** 941 (1970); J.S. Heyl and L. Hernquist, J. Phys. A: Math. Gen. **30** 6485 (1997); V.A. De Lorenci, R. Klippert, M. Novello and J.M. Salim, Phys. Lett. B **482** 134 (2000).
[13] M.H. Thoma, Europhys. Lett. **52** 498 (2000).
[14] D. Mihalas and B. Weibel–Mihalas, *Foundation of Radiation Hydrodynamics* (Dover Publications, 1999).
[15] V.I. Karpman, Plasma Phys. **13** 477 (1971); V.I. Karpman, Phys. Scr. **11** 263 (1975); Karpman V. I. , *Nonlinear waves in dispersive media* (Pergamon Press, Oxford, 1975).
[16] M. Marklund, G. Brodin and L. Stenflo, to appear in Phys. Rev. Lett (http://www.arxiv.org/pdf/astro-ph/0305269).
[17] J.T. Mendonça, *Theory of Photon Acceleration* (Institute of Physics Publishing, Bristol, 2001).